

NEARLY CLOAKING THE FULL MAXWELL EQUATIONS

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ABSTRACT. The approximate cloaking is investigated for time-harmonic Maxwell's equations via the approach of transformation optics. The problem is reduced to certain boundary effect estimates due to an inhomogeneous electromagnetic inclusion with an asymptotically small support but an arbitrary content enclosed by a thin high-conducting layer. Sharp estimates are established in terms of the asymptotic parameter, which are independent of the material tensors of the small electromagnetic inclusion. The result implies that the 'blow-up-a-small-region' construction via the transformation optics approach yields a near-cloak for the electromagnetic waves. A novelty lies in the fact that the geometry of the cloaking construction of this work can be very general. Moreover, by incorporating the conducting layer developed in the present paper right between the cloaked region and the cloaking region, arbitrary electromagnetic contents can be nearly cloaked. Our mathematical technique extends the general one developed in [30] for nearly cloaking scalar optics. In order to investigate the approximate electromagnetic cloaking for general geometries with arbitrary cloaked contents, new techniques and analysis tools must be developed for this more challenging vector optics case.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

This paper is concerned with invisibility cloaking for electromagnetic (EM) waves via the approach of transformation optics [19, 20, 27, 39], which is a rapidly growing scientific field with many potential applications. We refer to [9, 17, 18, 36, 42, 43] and the references therein for discussions of the recent progress on both the theory and experiments.

Let D and Ω be two bounded smooth domains in \mathbb{R}^3 , and $D \Subset \Omega$ with D convex containing the origin and Ω simply connected. Denote

$$D_\rho := \{\rho x; x \in D\} \quad \text{for } \rho \in \mathbb{R}_+.$$

Let $\varepsilon(x) = (\varepsilon^{ij}(x))_{i,j=1}^3$, $\mu(x) = (\mu^{ij}(x))_{i,j=1}^3$ and $\sigma(x) = (\sigma^{ij}(x))_{i,j=1}^3$, $x \in \Omega$ be real symmetric-matrix-valued functions, which are bounded in the sense that

$$c|\xi|^2 \leq \sum_{i,j=1}^3 \varepsilon^{ij}(x) \xi_i \xi_j \leq C|\xi|^2, \quad c|\xi|^2 \leq \sum_{i,j=1}^3 \mu^{ij}(x) \xi_i \xi_j \leq C|\xi|^2 \quad (1.1)$$

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and

$$0 \leq \sum_{i,j=1}^3 \sigma^{ij}(x) \xi_i \xi_j \leq C |\xi|^2, \quad (1.2)$$

for all $x \in \Omega$ and $\xi = (\xi_i)_{i=1}^3 \in \mathbb{R}^3$. Here c and C are two generic positive constants whose meanings should be clear from the contexts. Physically, the functions ε , μ and σ respectively stand for the electric permittivity, magnetic permeability and conductivity tensors of a *regular* EM medium occupying Ω .

Let $0 < \rho < 1$ be a small parameter. Assume that there exists an orientation-preserving bi-Lipschitz mapping $F_\rho : \overline{\Omega} \setminus D_\rho \rightarrow \overline{\Omega} \setminus D$, such that

$$F_\rho(\overline{\Omega} \setminus D_\rho) = \overline{\Omega} \setminus D, \quad F_\rho|_{\partial\Omega} = \text{Identity}. \quad (1.3)$$

Set

$$F(x) = \begin{cases} F_\rho(x), & x \in \Omega \setminus \overline{D}_\rho, \\ \frac{x}{\rho}, & x \in D_\rho. \end{cases} \quad (1.4)$$

Define an EM medium inside $\Omega \setminus \overline{D}$ as follows

$$\varepsilon_c^\rho(x) = F_* \varepsilon_0(x), \quad \mu_c^\rho(x) = F_* \mu_0(x), \quad \sigma_c^\rho(x) = 0, \quad x \in \Omega \setminus \overline{D}, \quad (1.5)$$

where $\varepsilon_0^{ij} = \delta^{ij}$ and $\mu_0^{ij} = \delta^{ij}$ with δ^{ij} the Kronecker delta function denote the EM parameter tensors of the homogeneous free space. The *push-forward* in (1.5) is defined by

$$F_* m(x) := \frac{DF(y) \cdot m(y) \cdot DF(y)^T}{|\det(DF)(y)|} \Big|_{y=F^{-1}(x)}, \quad x \in \Omega \setminus \overline{D} \quad (1.6)$$

where $m(y)$, $y \in \Omega \setminus \overline{D}_\rho$, denotes an EM parameter in $\Omega \setminus \overline{D}_\rho$, such as ε , μ or σ , and DF represents the Jacobian matrix of the transformation F . Also, one may rewrite (1.5) as

$$(\Omega \setminus \overline{D}; \varepsilon_c^\rho, \mu_c^\rho) = F_*(\Omega \setminus \overline{D}_\rho; \varepsilon_0, \mu_0) = (F(\Omega \setminus \overline{D}_\rho); F_* \varepsilon_0, F_* \mu_0).$$

Similarly, set

$$(D \setminus \overline{D}_{1/2}; \varepsilon_l, \mu_l, \sigma_l) = F_*(D_\rho \setminus \overline{D}_{\rho/2}; \alpha_0 \varepsilon_0, \beta_0 \mu_0, \gamma_0 \rho^{-2} \delta), \quad (1.7)$$

where α_0, β_0 and γ_0 are positive constants, and

$$(D_{1/2}; \tilde{\varepsilon}_a, \tilde{\mu}_a, \tilde{\sigma}_a) = F_*(D_{\rho/2}; \varepsilon_a, \mu_a, \sigma_a), \quad (1.8)$$

with *arbitrary* but *regular* ε_a, μ_a and σ_a . Hence, we have an EM medium in Ω given by

$$\Omega; \tilde{\varepsilon}, \tilde{\mu}, \tilde{\sigma} = \begin{cases} \varepsilon_c^\rho, \mu_c^\rho, \sigma_c^\rho & \text{in } \Omega \setminus \overline{D}, \\ \varepsilon_l, \mu_l, \sigma_l & \text{in } D \setminus \overline{D}_{1/2}, \\ \tilde{\varepsilon}_a, \tilde{\mu}_a, \tilde{\sigma}_a & \text{in } D_{1/2}. \end{cases} \quad (1.9)$$

The time-harmonic EM waves propagating in Ω is governed by the following Maxwell equations

$$\begin{cases} \nabla \wedge \tilde{E}_\rho - i\omega\tilde{\mu}\tilde{H}_\rho = 0 \\ \nabla \wedge \tilde{H}_\rho + i\omega(\tilde{\varepsilon} + i\frac{\tilde{\sigma}}{\omega})\tilde{E}_\rho = 0 \end{cases} \quad \text{in } \Omega, \quad (1.10)$$

where $\tilde{E}_\rho \in \mathbb{C}^3$ and $\tilde{H}_\rho \in \mathbb{C}^3$ denote, respectively, the electric and magnetic fields, and $\omega \in \mathbb{R}_+$ denotes the frequency.

Introduce the boundary operator $\tilde{\Lambda}_\rho$ which maps the tangential component of $\tilde{E}_\rho|_{\partial\Omega}$ to that of $\tilde{H}_\rho|_{\partial\Omega}$, i.e.,

$$\tilde{\Lambda}_\rho(\nu \wedge \tilde{E}_\rho|_{\partial\Omega}) = \nu \wedge \tilde{H}_\rho|_{\partial\Omega} : TH_{\text{Div}}^{-1/2}(\partial\Omega) \rightarrow TH_{\text{Div}}^{-1/2}(\partial\Omega), \quad (1.11)$$

where ν denotes the outward unit normal vector to $\partial\Omega$, and

$$TH_{\text{Div}}^{-1/2}(\partial\Omega) = \left\{ U \in TH^{-1/2}(\partial\Omega) \mid \text{Div}(U) \in H^{-1/2}(\partial\Omega) \right\},$$

with Div the surface divergence operator on $\partial\Omega$, $TH^s(\partial\Omega)$ the subspace of all those $V \in (H^s(\partial\Omega))^3$ which are orthogonal to ν and $H^s(\cdot)$ the usual L^2 -based Sobolev space of order $s \in \mathbb{R}$. Note that if Γ is the smooth boundary of a bounded domain in \mathbb{R}^3 , then $H^s(\Gamma)$ and hence $TH^s(\Gamma)$ is well defined for $|s| \leq 2$; see [21] and [29]. In (1.11), $\tilde{E}_\rho \in H(\nabla \wedge; \Omega)$ is the unique solution to the Maxwell equations (1.10) associated with the following boundary condition

$$\nu \wedge \tilde{E}_\rho|_{\partial\Omega} = \psi \in TH_{\text{Div}}^{-1/2}(\partial\Omega), \quad (1.12)$$

where

$$H(\nabla \wedge; \Omega) = \{U \in (L^2(\Omega))^3 \mid \nabla \wedge U \in (L^2(\Omega))^3\}.$$

In fact, $\tilde{\Lambda}_\rho$ is also known as the *admittance map* in the literature.

We further introduce the ‘free-space’ admittance map as follows. Let $E_0 \in H(\nabla \wedge; \Omega)$ and $H_0 \in H(\nabla \wedge; \Omega)$ be solutions to

$$\begin{cases} \nabla \wedge E_0 - i\omega\mu_0 H_0 = 0 & \text{in } \Omega, \\ \nabla \wedge H_0 + i\omega\varepsilon_0 E_0 = 0 & \text{in } \Omega, \\ \nu \wedge E_0|_{\partial\Omega} = \psi \in TH_{\text{Div}}^{-1/2}(\partial\Omega), \end{cases} \quad (1.13)$$

It is assumed that ω is not an EM eigenvalue to the Maxwell equations (1.13); namely, if $\psi = 0$, then one must have $E_0 = H_0 = 0$ for (1.13). Hence we have a well-defined admittance map

$$\Lambda_0(\psi) = \nu \wedge H_0|_{\partial\Omega} : TH_{\text{Div}}^{-1/2}(\partial\Omega) \rightarrow TH_{\text{Div}}^{-1/2}(\partial\Omega), \quad (1.14)$$

where $H_0 \in H(\nabla \wedge; \Omega)$ is the unique solution to (1.13). We refer to [35] and [26] for studies on the well-posedness of the Maxwell equations in the function setting introduced above.

We are ready to state the main result of this paper.

Theorem 1.1. *Suppose ω is not an EM eigenvalue of the free-space Maxwell equations (1.13). Let $\tilde{\Lambda}_\rho$ be the boundary admittance map in (1.11) associated with (1.10), where the EM parameter tensors are given by (1.5)–(1.9). Let Λ_0 be the “free” admittance map in (1.14) associated with (1.13). Then there exists a positive constant ρ_0 such that for any $\rho < \rho_0$,*

$$\|\tilde{\Lambda}_\rho - \Lambda_0\|_{\mathcal{L}(TH_{\text{Div}}^{-1/2}(\partial\Omega), TH_{\text{Div}}^{-1/2}(\partial\Omega))} \leq C\rho^3, \quad (1.15)$$

where C is a positive constant dependent only on $\rho_0, \omega, \alpha_0, \beta_0, \gamma_0$ and D, Ω , but completely independent of $\rho, \tilde{\varepsilon}_a, \tilde{\mu}_a$ and $\tilde{\sigma}_a$.

Before we proceed to prove the result, some general remarks about the significance of the result are in order.

Theorem 1.1 states that the transformation medium $(\Omega \setminus \overline{D}; \varepsilon_c^\rho, \mu_c^\rho, \sigma_c^\rho)$ together with the conducting layer $(D \setminus \overline{D}_{1/2}; \varepsilon_l, \mu_l, \sigma_l)$ in (1.9) produces an approximate invisibility cloaking device which nearly cloaks an arbitrary target medium $(D_{1/2}; \tilde{\varepsilon}_a, \tilde{\mu}_a, \tilde{\sigma}_a)$ located in the innermost region. Indeed, in the limiting case with $\rho = 0$ within spherical geometry, namely Ω and D are both Euclidean balls, (1.9) without the conducting layer yields the perfect invisibility cloaking construction in [39, 16]. That is, $\Lambda_\rho = \Lambda_0$ for $\rho = 0$, and hence by using the exterior boundary measurements encoded in Λ_ρ , one cannot “see” the inside object. However, it is widely known that the construction employs singular materials; that is, the material tensors ε_c^ρ and μ_c^ρ in the limiting case $\rho = 0$ possess degenerate singularities (cf. [16]). This presents a great challenge for both theoretical analysis and practical fabrications. In order to avoid the singular structure/materials, several regularized constructions have been developed. In [14, 15, 40], a truncation of singularities has been introduced. In [24, 25, 31], the ‘blow-up-a-point’ transformation in [20, 27, 39] has been regularized to become the ‘blow-up-a-small-region’ transformation. Nevertheless, as pointed out in [23], the truncation-of-singularity construction and the blow-up-a-small-region construction are equivalent to each other. Hence, our present study focuses on the blow-up-a-small-region construction; that is, F_ρ is used to blow up D_ρ of the relative size $\rho \ll 1$ for constructing the cloaking medium in (1.5). Theorem 1.1 states that our cloaking construction (1.9) yields an approximate cloaking device within ρ^3 -accuracy of the perfect cloak. Furthermore, an arbitrary content can be nearly cloaked.

Due to its practical importance, the approximate cloaking has recently been extensively studied. In [25, 2], approximate cloaking schemes were developed for EIT (electric impedance tomography) which might be regarded as optics at zero frequency. In [3, 4, 24, 28, 30, 31], various near-cloaking schemes were presented for scalar waves governed by the Helmholtz equation. In [32], a similar construction to (1.9) was developed for the full Maxwell equations. However, the study in [32] was only conducted for spherical geometry and the uniform cloaked content; that is, both Ω and D in (1.9) were assumed to be Euclidean balls and the medium parameters $\tilde{\varepsilon}_a, \tilde{\mu}_a$ and $\tilde{\sigma}_a$

were all constants multiple of the identity matrix. Under these assumptions, the Fourier-Bessel technique can be used to derive the analytic series expansions of the EM fields [32]. To our best knowledge, Theorem 1.1 is the first result for nearly cloaking the full Maxwell equations with general geometry and arbitrary cloaked contents. In order to assess the near-cloaking construction, the study is shown to be reduced to the boundary effect estimate due to an inhomogeneous electromagnetic inclusion with an asymptotically small support but an arbitrary content enclosed by a thin conducting layer with an asymptotically high conductivity tensor. The new structures of our problem require novel mathematical arguments. Our mathematical analysis uses the general strategy developed in [30] for nearly cloaking the scalar Helmholtz equation. However, new technique and estimates must be developed to deal with the general vector Maxwell equations. Finally, we point out that incorporating a damping mechanism by a conducting layer into the near-cloaking construction (1.9) is necessary for achieving successful near-cloak. In fact, it has been shown in [32] that no matter how small the regularization parameter ρ is, there always exist cloak-busting inclusions. Moreover, the result in [32] for the special case within spherical geometry and uniform cloaked contents confirms that our estimate in Theorem 1.1 is sharp.

For noninvasive EM detections, an related inverse problem is to extract physical information of the interior object, namely, $\tilde{\varepsilon}$, $\tilde{\mu}$ and $\tilde{\sigma}$ from the knowledge of the exterior EM measurements encoded into the boundary operator $\tilde{\Lambda}_\rho$. We refer the reader to [37] and [38] and the references therein for results on uniqueness and stability of this important inverse problem.

The rest of the paper is organized as follows. In Section 2, we present the proof of Theorem 1.1. Section 3 is devoted to the proof of a key lemma that was needed in the proof of Theorem 1.1.

2. PROOF OF THE MAIN THEOREM

2.1. Proof of Theorem 1.1. We first present a lemma with some key ingredients of the transformation optics, the proofs of which are available in [32].

Lemma 2.1. *Suppose that $E \in H(\nabla \wedge; \Omega)$ and $H \in H(\nabla \wedge; \Omega)$ are EM fields satisfying*

$$\begin{aligned} \nabla \wedge E - i\omega\mu H &= 0 & \text{in } \Omega, \\ \nabla \wedge H + i\omega \left(\varepsilon + i\frac{\sigma}{\omega} \right) E &= 0 & \text{in } \Omega, \end{aligned}$$

where $(\Omega; \varepsilon, \mu, \sigma)$ is a regular EM medium. Let $x' = \mathcal{F}(x) : \Omega \rightarrow \Omega$ be a bi-Lipschitz and orientation-preserving mapping such that $\mathcal{F}|_{\partial\Omega} = \text{Identity}$. Define the pull-back EM fields by

$$\begin{aligned} E' &= (\mathcal{F}^{-1})^* E := (D\mathcal{F})^{-T} E \circ \mathcal{F}^{-1}, \\ H' &= (\mathcal{F}^{-1})^* H := (D\mathcal{F})^{-T} H \circ \mathcal{F}^{-1}. \end{aligned}$$

Then, $E' \in H(\nabla' \wedge; \Omega)$ and $H' \in H(\nabla' \wedge; \Omega)$. In addition the following identities hold

$$\begin{aligned} \nabla' \wedge E' &= i\omega\mu' H' & \text{in } \Omega \\ \nabla' \wedge H' &= -i\omega \left(\varepsilon' + i\frac{\sigma'}{\omega} \right) E' & \text{in } \Omega, \end{aligned}$$

where $\nabla' \wedge$ denote the curl operator in the x' -coordinates, and ε' , μ' and σ' are the push-forwards of ε , μ and σ via \mathcal{F} , namely,

$$(\Omega; \varepsilon', \mu', \sigma') = \mathcal{F}_*(\Omega; \varepsilon, \mu, \sigma).$$

Particularly, if one lets Λ and Λ' denote the admittance maps associated with (E, H) and (E', H') , respectively. Then

$$\Lambda = \Lambda'.$$

Next, for the EM fields $(\tilde{E}_\rho, \tilde{H}_\rho)$ in (1.10) associated with the boundary condition (1.12), we let

$$E_\rho = F^* \tilde{E}_\rho \quad \text{and} \quad H_\rho = F^* \tilde{H}_\rho. \quad (2.1)$$

Then by Lemma 2.1, it is seen that $E_\rho \in H(\nabla \wedge; \Omega)$ and $H_\rho \in H(\nabla \wedge; \Omega)$ which satisfy the following Maxwell equations,

$$\begin{cases} \nabla \wedge E_\rho - i\omega\mu_\rho H_\rho = 0 & \text{in } \Omega, \\ \nabla \wedge H_\rho + i\omega \left(\varepsilon_\rho + i\frac{\sigma_\rho}{\omega} \right) E_\rho = 0 & \text{in } \Omega, \\ \nu \wedge E_\rho|_{\partial\Omega} = \psi \in TH_{\text{Div}}^{-1/2}(\partial\Omega), \end{cases} \quad (2.2)$$

where

$$\Omega; \varepsilon_\rho, \mu_\rho, \sigma_\rho = \begin{cases} \varepsilon_0, \mu_0, 0 & \text{in } \Omega \setminus D_\rho, \\ \alpha_0 \varepsilon_0, \beta_0 \rho^2 \mu_0, \gamma_0 \rho^{-2} \delta & \text{in } D_\rho \setminus \overline{D}_{\rho/2}, \\ \varepsilon_a, \mu_a, \sigma_a & \text{in } D_{\rho/2}. \end{cases} \quad (2.3)$$

Furthermore, by Lemma (2.1),

$$\Lambda_\rho = \tilde{\Lambda}_\rho \quad (2.4)$$

where Λ_ρ is the admittance map associated with the EM fields (E_ρ, H_ρ) in (2.2). Hence, in order to prove Theorem 1.1, it suffices to show the following result.

Theorem 2.1. *Suppose ω is not an EM eigenvalue of the free-space Maxwell equations (1.13). Let $(E_0, H_0) \in H(\nabla \wedge; \Omega) \wedge H(\nabla \wedge; \Omega)$ and $(E_\rho, H_\rho) \in H(\nabla \wedge; \Omega) \wedge H(\nabla \wedge; \Omega)$ be solutions to (1.13) and (2.2), respectively. Then there exists a positive constant ρ_0 such that for any $\rho < \rho_0$,*

$$\|\nu \wedge H_\rho - \nu \wedge H_0\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} \leq C\rho^3 \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)}, \quad (2.5)$$

where C is a positive constant dependent only on $\rho_0, \omega, \alpha_0, \beta_0, \gamma_0$ and D, Ω , but completely independent of $\rho, \varepsilon_a, \mu_a$ and σ_a and ψ .

Remark 2.1. Qualitatively speaking, Theorem 2.1 states that the boundary EM effects due to an inhomogeneous EM inclusion supported in a small region D_ρ with arbitrary contents in $D_{\rho/2}$ but enclosed by a thin layer of high conducting medium in $D_\rho \setminus \overline{D}_{\rho/2}$ is also small. In the literature, there are extensive studies on the scattering estimates due to small EM scatterers (see [5, 7, 8]), and also the closely related low-frequency asymptotics of EM scattering (i.e., the Rayleigh approximation; see [6, 13, 33, 35]). However, the EM inclusions of the aforementioned studies have fixed contents, and indeed they are either perfectly conducting obstacles or EM mediums with piecewise constant material parameters. In Theorem 2.1, the small EM inclusion of our current study has peculiar structures, and the quantitative estimate (2.5) cannot be adapted from existing results in the literature.

The rest of the paper is devoted to the proof of Theorem 2.1. In order to simplify the exposition, we set

$$\alpha_0 = \beta_0 = \gamma_0 = 1.$$

We next derive three key lemmas.

Lemma 2.2. *The solutions of (1.13) and (2.2) satisfy*

$$\int_{D_\rho \setminus D_{\rho/2}} |E_\rho|^2 d\sigma_x \leq C\rho^2 \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} \|\nu \wedge (H_\rho - H_0)\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)}, \quad (2.6)$$

where C is a positive constant depending only on Ω .

Proof. Inner-producting both sides of the second equation in (2.2) by \overline{E}_ρ , and integrating by parts, we have

$$\begin{aligned} & \int_{\Omega} -i\omega \left(\varepsilon_\rho + i\frac{\sigma_\rho}{\omega} \right) E_\rho \cdot \overline{E}_\rho d\sigma_x = \int_{\Omega} (\nabla \wedge H_\rho) \cdot \overline{E}_\rho d\sigma_x \\ &= \int_{\Omega} H_\rho \cdot (\nabla \wedge \overline{E}_\rho) d\sigma_x - \int_{\partial\Omega} (\nu \wedge \overline{E}_\rho) \cdot H_\rho ds_x \\ &= \int_{\Omega} H_\rho \cdot (-i\omega \overline{\mu}_\rho \overline{H}_\rho) d\sigma_x + \int_{\partial\Omega} (\nu \wedge \overline{E}_\rho) \cdot [\nu \wedge (\nu \wedge H_\rho)] ds_x \end{aligned} \quad (2.7)$$

In (2.7), we have made use of the decomposition that

$$H_\rho|_{\partial\Omega} = (H_\rho|_{\partial\Omega})_t + \nu(H_\rho|_{\partial\Omega})_\nu,$$

where the tangential component $(H_\rho|_{\partial\Omega})_t$ is $-\nu \wedge (\nu \wedge (H_\rho|_{\partial\Omega}))$ and the normal component is $(H_\rho|_{\partial\Omega})_\nu$ is $\langle \nu, H_\rho|_{\partial\Omega} \rangle$. By taking the real parts of both sides of (2.7), we have

$$\begin{aligned} & \int_{D_{\rho/2}} \sigma_a E_\rho \cdot \overline{E}_\rho d\sigma_x + \rho^{-2} \int_{D_\rho \setminus D_{\rho/2}} |E_\rho|^2 d\sigma_x \\ &= \Re \int_{\partial\Omega} (\nu \wedge \overline{E}_\rho) \cdot [\nu \wedge (\nu \wedge H_\rho)] ds_x. \end{aligned} \quad (2.8)$$

On the other hand, it is straightforward to verify that

$$0 = \Re \int_{\partial\Omega} (\nu \wedge \overline{E}_0) \cdot [\nu \wedge (\nu \wedge H_0)] ds_x. \quad (2.9)$$

We shall make use of the following fact that the skew-symmetric bilinear form

$$\begin{aligned} \mathcal{B} : TH_{\text{Div}}^{-1/2}(\partial\Omega) \wedge TH_{\text{Div}}^{-1/2}(\partial\Omega) &\rightarrow \mathbb{C}, \\ (\mathbf{j}, \mathbf{m}) &\rightarrow \mathcal{B}(\mathbf{j}, \mathbf{m}) = \int_{\partial\Omega} \mathbf{j} \cdot (\mathbf{m} \wedge \nu) ds \end{aligned} \quad (2.10)$$

defines a non-degenerate duality product on $TH_{\text{Div}}^{-1/2}(\partial\Omega)$ (cf. [12]). Subtracting (2.9) from (2.8), one has

$$\begin{aligned} &\int_{D_{\rho/2}} \sigma_a E_\rho \cdot \overline{E}_\rho d\sigma_x + \rho^{-2} \int_{D_\rho \setminus D_{\rho/2}} |E_\rho|^2 d\sigma_x \\ &= \Re \int_{\partial\Omega} \overline{\psi} \cdot [\nu \wedge (\nu \wedge (H_\rho - H_0))] ds_x, \end{aligned}$$

which together with the duality (2.14) yields (2.6). \square

In the sequel, we let

$$\nu \wedge E_\rho^-(x) \quad (\text{resp. } \nu \wedge H_\rho^-(x)) \quad \text{on } \partial D_\rho$$

denote the tangential component of E_ρ (resp. H_ρ) on ∂D_ρ when one approaches ∂D_ρ from the interior of D_ρ . Similarly, we let

$$\nu \wedge E_\rho^+(x) \quad (\text{resp. } \nu \wedge H_\rho^+(x)) \quad \text{on } \partial D_\rho$$

denote the tangential component of E_ρ (resp. H_ρ) on ∂D_ρ when one approaches D_ρ from the exterior of D_ρ .

Lemma 2.3. *The solutions to (1.13) and (2.2) satisfy*

$$\begin{aligned} &\|(\nu \wedge E_\rho^-)(\rho \cdot)\|_{TH^{-1/2}(\partial D)}^2 \\ &\leq C \rho^{-1} \left| 1 + \omega^2 \rho^2 \left(1 + i \frac{\rho^{-2}}{\omega} \right) \right|^2 \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} \|\nu \wedge (H_\rho - H_0)\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)}, \end{aligned} \quad (2.11)$$

and hence by the transmission condition across ∂D_ρ

$$\begin{aligned} &\|(\nu \wedge E_\rho^+)(\rho \cdot)\|_{TH^{-1/2}(\partial D)}^2 \\ &\leq C \rho^{-1} \left| 1 + \omega^2 \rho^2 \left(1 + i \frac{\rho^{-2}}{\omega} \right) \right|^2 \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} \|\nu \wedge (H_\rho - H_0)\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)}, \end{aligned} \quad (2.12)$$

where C is a positive constant dependent only on D and Ω , but independent of ψ and ρ .

Proof. Clearly, it suffices to show that

$$\begin{aligned} & \|(\nu \wedge E_\rho^-)(\rho \cdot)\|_{(H^{-1/2}(\partial D))^3}^2 \\ & \leq C\rho^{-1} \left| 1 + \omega^2 \rho^2 \left(1 + i \frac{\rho^{-2}}{\omega} \right) \right|^2 \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} \|\nu \wedge (H_\rho - H_0)\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)}. \end{aligned} \quad (2.13)$$

Our proof begins with the following duality identity

$$\begin{aligned} & \|(\nu \wedge E_\rho^-)(\rho \cdot)\|_{(H^{-1/2}(\partial D))^3} \\ & = \sup_{\|\phi\|_{(H^{1/2}(\partial\Omega))^3} \leq 1} \left| \int_{\partial D} (\nu \wedge E_\rho^-)(\rho x) \cdot \phi(x) \, ds_x \right|. \end{aligned} \quad (2.14)$$

For any $\phi \in (H^{1/2}(\partial D))^3$, there exists $U \in (H^2(D))^3$ such that (see Remark 2.2)

- (i) $\nu \wedge U = 0$ on ∂D ,
- (ii) $\nu \wedge (\nu \wedge (\nabla \wedge U)) = \nu \wedge (\nu \wedge \phi)$ on ∂D ,
- (ii) $\|U\|_{(H^2(D))^3} \leq C\|\phi\|_{(H^{1/2}(\partial D))^3}$,
- (iii) $\nabla \cdot U = 0$ in $D \setminus D_{3/4}$ and $U = 0$ in $D_{1/2}$.

Then

$$\begin{aligned} & \int_{\partial D} (\nu \wedge E_\rho^-)(\rho x) \cdot \phi(x) \, ds_x \\ & = - \int_{\partial D} (\nu \wedge E_\rho^-)(\rho x) \cdot (\nu \wedge (\nu \wedge \phi))(x) \, ds_x \\ & = - \int_{\partial D} (\nu \wedge E_\rho^-)(\rho x) \cdot (\nu \wedge (\nu \wedge (\nabla \wedge U)))(x) \, ds_x \\ & = \int_{\partial D} (\nu \wedge E_\rho^-)(\rho x) \cdot (\nabla \wedge U)(x) \, ds_x. \end{aligned} \quad (2.15)$$

For $y \in D_\rho$, let

$$x := \frac{y}{\rho} \in D.$$

Set

$$E(x) := E_\rho(\rho x) = E_\rho(y), \quad x \in D.$$

Since

$$\begin{aligned} & \nabla_y \wedge E_\rho(y) - i\omega H_\rho(y) = 0, \\ & \nabla_y \wedge H_\rho(y) + i\omega \left(1 + i \frac{\rho^{-2}}{\omega} \right) E_\rho(y) = 0, \end{aligned} \quad (2.16)$$

for $y \in D_\rho \setminus \overline{D}_{\rho/2}$, it is easily verified that

$$\begin{aligned}\nabla_x \wedge E(x) &= i\omega\rho H(x), \\ \nabla_x \wedge H(x) &= -i\omega\rho \left(1 + i\frac{\rho^{-2}}{\omega}\right) E(x), \quad x \in D \setminus \overline{D}_{1/2}.\end{aligned}\tag{2.17}$$

Then, by (2.15)–(2.17), and Green's formula, we have

$$\begin{aligned}& \int_{\partial D} (\nu \wedge E_\rho^-)(\rho x) \cdot \phi(x) \, ds_x \\ &= \int_{\partial D} (\nu \wedge E^-)(x) \cdot (\nabla \wedge U)(x) \, ds_x \\ &= \int_{\partial D} (\nu \wedge E^-)(x) \cdot (\nabla \wedge U)(x) - (\nu \wedge U)(x) \cdot (\nabla \wedge E)(x) \, ds_x \\ &= \int_D E \cdot \Delta U - U \cdot \Delta E \, d\sigma_x.\end{aligned}\tag{2.18}$$

By (2.17), it is straightforward to show that

$$\Delta E(x) + \omega^2 \rho^2 \left(1 + i\frac{\rho^{-2}}{\omega}\right) E(x) = 0, \quad x \in D \setminus \overline{D}_{1/2}.\tag{2.19}$$

Therefore, from (2.18) and (2.19), it follows directly that

$$\begin{aligned}& \int_{\partial D} (\nu \wedge E_\rho^-)(\rho x) \cdot \phi(x) \, ds_x \\ &= \int_D E \cdot \Delta U - U \cdot \Delta E \, d\sigma_x \\ &= \left[1 + \omega^2 \rho^2 \left(1 + i\frac{\rho^{-2}}{\omega}\right)\right] \int_{D \setminus D_{1/2}} E \cdot (U + \Delta U) \, d\sigma_x,\end{aligned}\tag{2.20}$$

hence

$$\begin{aligned}& \left| \int_{\partial D} (\nu \wedge E_\rho^-)(\rho x) \cdot \phi(x) \, ds_x \right| \\ &\leq \left| 1 + \omega^2 \rho^2 \left(1 + i\frac{\rho^{-2}}{\omega}\right) \right| \|E\|_{L^2(D \setminus D_{1/2})} \|\phi\|_{(H^{1/2}(\partial D))^3}\end{aligned}\tag{2.21}$$

Using the relation

$$\|E\|_{L^2(D \setminus D_{1/2})} = \|E_\rho(\rho \cdot)\|_{L^2(D \setminus D_{1/2})} = \rho^{-3/2} \|E_\rho\|_{L^2(D_\rho \setminus D_{\rho/2})},$$

we have from (2.21) that

$$\begin{aligned}& \|(\nu \wedge E_\rho^-)(\rho \cdot)\|_{(H^{-1/2}(\partial D))^3} \\ &\leq \left| 1 + \omega^2 \rho^2 \left(1 + i\frac{\rho^{-2}}{\omega}\right) \right| \|E\|_{L^2(D \setminus D_{1/2})} \\ &\leq \rho^{-3/2} \left| 1 + \omega^2 \rho^2 \left(1 + i\frac{\rho^{-2}}{\omega}\right) \right| \|E_\rho\|_{L^2(D_\rho \setminus D_{\rho/2})},\end{aligned}$$

which together with (2.6) in Lemma 2.2 immediately implies (2.13).

The proof is now completed. \square

Remark 2.2. The auxiliary function U in the proof of Lemma 2.3 can be given as $\xi\tilde{U}$, where $\xi(x) \in C^\infty(\overline{D})$ is such that $0 \leq \xi \leq 1$, $\xi = 0$ in $D_{1/2}$ and $\xi = 1$ in $D \setminus D_{3/4}$, and $\tilde{U} = (\nabla \wedge)^2 V$ with $V \in (H^4(D))^3$ being the unique solution to the following auxiliary differential equation

$$\begin{cases} (\nabla \wedge)^4 V + V = 0 & \text{in } D, \\ \nu \wedge (\nabla \wedge)^2 V = 0 & \text{on } \partial D, \\ \nu \wedge (\nu \wedge (\nabla \wedge)^3 V) = \nu \wedge (\nu \wedge \phi) & \text{on } \partial D. \end{cases} \quad (2.22)$$

The unique existence of a solution to (2.23) can be shown by the standard way of using Riesz-Fredholm theory, and it could also follow from a more general theory presented in [22], Chapter XX.

Remark 2.3. It is noted by (2.19) that the Cartesian components of E satisfy the Helmholtz equation. Lemma 2.3 could also be proved component-wisely by following the proofs of Lemma 3.3 in [24] and Lemma 3.3 in [30].

The next lemma is of crucial importance and its proof will be given in Section 3.

Lemma 2.4. *Suppose ω is not an eigenvalue of the free-space Maxwell equations (1.13). Let $E_0 \in H(\nabla \wedge; \Omega)$ and $H_0 \in H(\nabla \wedge; \Omega)$ be the solutions to (1.13). Let $\tau \in \mathbb{R}_+$ and let*

$$\varphi \in TH_{\text{Div}}^{-1/2}(\partial D_\tau).$$

Consider the Maxwell equations

$$\begin{cases} \nabla \wedge E_\tau - i\omega H_\tau = 0 & \text{in } \Omega \setminus \overline{D}_\tau, \\ \nabla \wedge H_\tau + i\omega E_\tau = 0 & \text{in } \Omega \setminus \overline{D}_\tau, \\ \nu \wedge E_\tau = \varphi & \text{on } \partial D_\tau, \\ \nu \wedge E_\tau = \psi & \text{on } \partial \Omega. \end{cases} \quad (2.23)$$

Then there exists a constant $\tau_0 \in \mathbb{R}_+$ such that for any $\tau < \tau_0$,

$$\begin{aligned} & \|\nu \wedge (H_\tau - H_0)\|_{TH^{-1/2}(\partial \Omega)} \\ & \leq C \left(\tau^3 \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial \Omega)} + \tau^2 \|\varphi(\tau \cdot)\|_{TH^{-1/2}(\partial D)} \right), \end{aligned} \quad (2.24)$$

where C is a generic positive constant dependent only on τ_0, ω and Ω, D , but independent of τ and φ, ψ .

It should be emphasized that in the estimate (2.24), the norm for $\varphi(\tau \cdot)$ is $TH^{-1/2}(\partial D)$ though $\varphi(\tau \cdot) \in TH_{\text{Div}}^{-1/2}(\partial D)$, and in this sense, the estimate is “non-standard”.

We are in position to present the proof of Theorem 1.1.

Proof of Theorem 1.1. By taking $\tau = \rho$ and $\varphi = \nu \wedge E_\rho^+|_{\partial D_\rho}$ in Lemma 2.4, we have

$$\begin{aligned} & \|\nu \wedge (H_\rho - H_0)\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} \\ & \leq C_1 \left(\rho^3 \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)}^{-1/2} + \rho^2 \|(\nu \wedge E_\rho^+)(\rho \cdot)\|_{TH^{-1/2}(\partial D)} \right). \end{aligned} \quad (2.25)$$

Next, by Lemma 2.3, we have for $\epsilon > 0$

$$\begin{aligned} & \|(\nu \wedge E_\rho^+)(\rho \cdot)\|_{TH^{-1/2}(\partial D)} \\ & \leq C_2 \rho^{-1/2} \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)}^{1/2} \|\nu \wedge (H_\rho - H_0)\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)}^{1/2} \\ & \leq C_2 \rho^{-1/2} \left(\frac{\rho^{3/2}}{4\epsilon} \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} + \frac{\epsilon}{\rho^{3/2}} \|\nu \wedge (H_\rho - H_0)\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} \right). \end{aligned} \quad (2.26)$$

From (2.25) and (2.26), we further have

$$\begin{aligned} & \|\nu \wedge (H_\rho - H_0)\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} \leq C_1 \rho^3 \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} \\ & + \frac{1}{4} C_1 C_2 \frac{\rho^3}{\epsilon} \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} + C_1 C_2 \epsilon \|\nu \wedge (H_\rho - H_0)\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)}. \end{aligned} \quad (2.27)$$

By choosing ϵ such that $C_1 C_2 \epsilon < 1/2$, we see immediately from (2.27) that

$$\|\nu \wedge (H_\rho - H_0)\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} \leq C \rho^3 \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)},$$

which completes the proof. \square

3. PROOF OF LEMMA 2.4

We present the proof of Lemma 2.4 which is crucial for the proof of our main theorem.

Set

$$\tilde{E}_\tau = E_\tau - E_0, \quad \tilde{H}_\tau = H_\tau - H_0.$$

It is straightforward to verify that

$$\begin{cases} \nabla \wedge \tilde{E}_\tau - i\omega \tilde{H}_\tau = 0 & \text{in } \Omega \setminus \overline{D}_\tau, \\ \nabla \wedge \tilde{H}_\tau + i\omega \tilde{E}_\tau = 0 & \text{in } \Omega \setminus \overline{D}_\tau, \\ \nu \wedge \tilde{E}_\tau = \varphi - \nu \wedge E_0 & \text{on } \partial D_\tau, \\ \nu \wedge \tilde{E}_\tau = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.1)$$

Obviously, in order to show (2.24), it suffices to show

$$\|\nu \wedge \tilde{H}_\tau\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} \leq C \left(\tau^3 \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} + \tau^2 \|\varphi(\tau \cdot)\|_{TH^{-1/2}(\partial D)} \right). \quad (3.2)$$

In order to prove (3.2), we let

$$\tilde{E}_\tau = U_\tau - \tilde{U}_\tau \quad \text{and} \quad \tilde{H}_\tau = V_\tau - \tilde{V}_\tau, \quad (3.3)$$

where $(U_\tau, V_\tau) \in H_{loc}(\nabla \wedge; \mathbb{R}^3 \setminus \overline{D}_\tau) \wedge H_{loc}(\nabla \wedge; \mathbb{R}^3 \setminus \overline{D}_\tau)$ are scattering solutions to

$$\begin{aligned} \nabla \wedge U_\tau - i\omega V_\tau &= 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}_\tau, \\ \nabla \wedge V_\tau + i\omega U_\tau &= 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}_\tau, \\ \nu \wedge U_\tau &= \varphi - \nu \wedge E_0 & \text{on } \partial D_\tau, \end{aligned} \quad (3.4)$$

$$\lim_{|x| \rightarrow +\infty} |x| \left| (\nabla \wedge U_\tau)(x) \wedge \frac{x}{|x|} - i\omega U_\tau(x) \right| = 0,$$

and $(\tilde{U}_\tau, \tilde{V}_\tau) \in H(\nabla \wedge; \Omega \setminus \overline{D}_\tau) \wedge H(\nabla \wedge; \Omega \setminus \overline{D}_\tau)$ are solutions to

$$\begin{aligned} \nabla \wedge \tilde{U}_\tau - i\omega \tilde{V}_\tau &= 0 & \text{in } \Omega \setminus \overline{D}_\tau, \\ \nabla \wedge \tilde{V}_\tau + i\omega \tilde{U}_\tau &= 0 & \text{in } \Omega \setminus \overline{D}_\tau, \\ \nu \wedge \tilde{U}_\tau &= \nu \wedge U_\tau & \text{on } \partial \Omega, \\ \nu \wedge \tilde{U}_\tau &= 0 & \text{on } \partial D_\tau. \end{aligned} \quad (3.5)$$

We shall show the following two lemmas, which immediately imply (3.2)

Lemma 3.1. *Let $(U_\tau, V_\tau) \in H_{loc}(\nabla \wedge; \mathbb{R}^3 \setminus \overline{D}_\tau) \wedge H_{loc}(\nabla \wedge; \mathbb{R}^3 \setminus \overline{D}_\tau)$ be scattering solutions to (3.4). Then there exists $\tau_0 \in \mathbb{R}_+$ such that for any $\tau < \tau_0$*

$$\|U_\tau\|_{(C^2(\partial\Omega))^3} \leq C \left(\tau^3 \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} + \tau^2 \|\varphi(\tau \cdot)\|_{TH^{-1/2}(\partial D)} \right), \quad (3.6)$$

and

$$\|V_\tau\|_{(C^2(\partial\Omega))^3} \leq C \left(\tau^3 \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} + \tau^2 \|\varphi(\tau \cdot)\|_{TH^{-1/2}(\partial D)} \right), \quad (3.7)$$

where C is a positive constant dependent only on τ_0, ω and Ω, D , but independent of τ and φ, ψ .

Lemma 3.2. *Let $(\tilde{U}_\tau, \tilde{V}_\tau) \in H(\nabla \wedge; \Omega \setminus \overline{D}_\tau) \wedge H(\nabla \wedge; \Omega \setminus \overline{D}_\tau)$ be solutions to (3.5). Then there exists $\tau_0 \in \mathbb{R}_+$ such that for any $\tau < \tau_0$*

$$\|\nu \wedge \tilde{V}_\tau\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} \leq C \left(\tau^3 \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} + \tau^2 \|\varphi(\tau \cdot)\|_{TH^{-1/2}(\partial D)} \right), \quad (3.8)$$

where C is a positive constant dependent only on τ_0, ω and Ω, D , but independent of τ and φ, ψ .

We next present the proof of Lemma 3.1, which may be further divided into the following two propositions.

Proposition 3.1. *Let $(U_{\tau,1}, V_{\tau,1}) \in H_{loc}(\nabla \wedge; \mathbb{R}^3 \setminus \overline{D}_\tau) \wedge H_{loc}(\nabla \wedge; \mathbb{R}^3 \setminus \overline{D}_\tau)$ be solutions to*

$$\begin{aligned} \nabla \wedge U_{\tau,1} - i\omega V_{\tau,1} &= 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}_\tau, \\ \nabla \wedge V_{\tau,1} + i\omega U_{\tau,1} &= 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}_\tau, \\ \nu \wedge U_{\tau,1} &= \nu \wedge E_0 & \text{on } \partial D_\tau, \end{aligned} \quad (3.9)$$

$$\lim_{|x| \rightarrow +\infty} |x| \left| (\nabla \wedge U_{\tau,1})(x) \wedge \frac{x}{|x|} - i\omega U_{\tau,1}(x) \right| = 0,$$

Then there exists $\tau_0 \in \mathbb{R}_+$ such that for any $\tau < \tau_0$

$$\|U_{\tau,1}\|_{(C^2(\partial\Omega))^3} \leq C\tau^3 \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} \quad (3.10)$$

and

$$\|V_{\tau,1}\|_{(C^2(\partial\Omega))^3} \leq C\tau^3 \|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)}, \quad (3.11)$$

where C is a positive constant dependent only on τ_0, ω and Ω, D , but independent of τ and φ, ψ .

Proposition 3.2. Let $(U_{\tau,2}, V_{\tau,2}) \in H_{loc}(\nabla \wedge; \mathbb{R}^3 \setminus \overline{D}_\tau) \wedge H_{loc}(\nabla \wedge; \mathbb{R}^3 \setminus \overline{D}_\tau)$ be solutions to

$$\begin{aligned} \nabla \wedge U_{\tau,2} - i\omega V_{\tau,2} &= 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}_\tau, \\ \nabla \wedge V_{\tau,2} + i\omega U_{\tau,2} &= 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}_\tau, \\ \nu \wedge U_{\tau,2} &= \varphi & \text{on } \partial D_\tau, \end{aligned} \quad (3.12)$$

$$\lim_{|x| \rightarrow +\infty} |x| \left| (\nabla \wedge U_{\tau,2})(x) \wedge \frac{x}{|x|} - i\omega U_{\tau,2}(x) \right| = 0,$$

Then there exists $\tau_0 \in \mathbb{R}_+$ such that for any $\tau < \tau_0$

$$\|U_{\tau,2}\|_{(C^2(\partial\Omega))^3} \leq C\tau^2 \|\varphi(\tau \cdot)\|_{TH^{-1/2}(\partial D)} \quad (3.13)$$

and

$$\|V_{\tau,2}\|_{(C^2(\partial\Omega))^3} \leq C\tau^2 \|\varphi(\tau \cdot)\|_{TH^{-1/2}(\partial D)}, \quad (3.14)$$

where C is a positive constant dependent only on τ_0, ω and Ω, D , but independent of τ and φ, ψ .

Proof of Proposition 3.1. We first note that the solutions E_0 and H_0 to (1.13) are smooth inside Ω , and also by the local regularity estimate we have

$$\|E_0\|_{(C^1(\overline{D}))^3} \leq C\|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)} \text{ and } \|H_0\|_{(C^1(\overline{D}))^3} \leq C\|\psi\|_{TH_{\text{Div}}^{-1/2}(\partial\Omega)}, \quad (3.15)$$

where C is constant depending only on Ω, D and ω . Since $\nu \wedge E_0$ is smooth on ∂D_τ , we know both $U_{\tau,1}$ and $V_{\tau,1}$ are strong solutions which belong to $(C^1(\mathbb{R}^3 \setminus \overline{D}_\tau) \cap C^{0,\alpha}(\mathbb{R}^3 \setminus D_\tau))^3$ with $0 < \alpha < 1$ (see [10, 11]). By a completely similar argument to the proof of Corollary 3.2 in [8], which is based on the low frequency asymptotics in [13], one can show (3.10) and (3.11). \square

Proof of Proposition 3.2. We make use of the layer potential technique to show the lemma. To that end, we let

$$G(x, y) := \frac{1}{4\pi} \frac{e^{i\omega|x-y|}}{|x-y|} \quad \text{and} \quad G_0(x, y) := \frac{1}{4\pi} \frac{1}{|x-y|}, \quad x, y \in \mathbb{R}^3; \quad x \neq y.$$

Furthermore, we introduce the following vector boundary layer potential operators \mathbf{M}_Γ and \mathbf{M}_Γ^0 ,

$$(\mathbf{M}_\Gamma \mathbf{a})(x) := 2 \int_\Gamma \nu(x) \wedge [\nabla_x \wedge (\mathbf{a}(y)G(x, y))] ds_y, \quad x \in \Gamma \quad (3.16)$$

and

$$(\mathbf{M}_\Gamma^0 \mathbf{a})(x) := 2 \int_\Gamma \nu(x) \wedge [\nabla_x \wedge (\mathbf{a}(y) G_0(x, y))] ds_y, \quad x \in \Gamma, \quad (3.17)$$

where \mathbf{a} is a tangential vector field on Γ . We refer to [10, 34, 35, 41] for related mapping properties of the above introduced operators.

We make use of the following *ansatz* of the EM fields to (3.12),

$$U_{\tau,2}(x) = \nabla_x \wedge \int_{\partial D_\tau} G(x, y) \mathbf{a}(y) ds_y, \quad x \in \mathbb{R}^3 \setminus \overline{D}_\tau, \quad (3.18)$$

$$\begin{aligned} V_{\tau,2}(x) &= \frac{1}{i\omega} \nabla_x \wedge U_{\tau,2}(x) = \frac{\omega}{i} \int_{\partial D_\tau} G(x, y) \mathbf{a}(y) ds_y \\ &\quad + \frac{1}{i\omega} \int_{\partial D_\tau} \nabla_x G(x, y) \text{Div} \mathbf{a}(y) ds_y, \quad x \in \mathbb{R}^3 \setminus \overline{D}_\tau, \end{aligned} \quad (3.19)$$

where $\mathbf{a} \in TH_{\text{Div}}^{-1/2}(\partial D_\tau)$. One needs first show the uniform well-posedness of the Maxwell equations (3.12). That is, for any fixed $\omega \in \mathbb{R}_+$, there exists $\tau_0 \in \mathbb{R}_+$ sufficiently small such that when $\tau < \tau_0$, there exists a unique $\mathbf{a} \in TH_{\text{Div}}^{-1/2}(\partial D_\tau)$ such that $U_{\tau,2}$ and $V_{\tau,2}$ given in (3.18) and (3.19) are solutions to (3.12). However, we shall not show this in the sequel, and instead we shall directly estimate \mathbf{a} assuming its existence which will then give the desired estimates (3.13) and (3.14). Nevertheless, we emphasize that it can be directly seen from our estimating of \mathbf{a} the uniform well-posedness of (3.12), or equivalently the unique existence of $\mathbf{a} \in TH_{\text{Div}}^{-1/2}(\partial D_\tau)$.

Next, by letting x approach ∂D_τ^+ , and using the mapping properties of $\mathbf{M}_{\partial D_\tau}$ and the jump properties of the vector potential operator $\mathbf{M}_{\partial D_\tau}$, one has

$$\mathbf{a}(x) + [\mathbf{M}_{\partial D_\tau} \mathbf{a}](x) = 2\varphi(x), \quad x \in \partial D_\tau. \quad (3.20)$$

We claim that for τ sufficiently small

$$\|\mathbf{a}(\tau \cdot)\|_{TH^{-1/2}(\partial D)} \leq C \|\varphi(\tau \cdot)\|_{TH^{-1/2}(\partial D)}, \quad (3.21)$$

where C is a generic constant independent of τ and φ . To that end, we let

$$\tilde{\mathbf{a}}(x') =: \mathbf{a}(\tau x'), \quad x' := \frac{x}{\tau} \in \partial D, \quad x \in \partial D_\tau.$$

By using change of variables,

$$x' = x/\tau \text{ and } y' = y/\tau \text{ for } x, y \in \partial D_\tau, \quad (3.22)$$

one can show

$$\begin{aligned} (\mathbf{M}_{\partial D_\tau} \mathbf{a})(x) &= (\mathbf{M}_{\partial D_\tau} \mathbf{a})(\tau x') \\ &= 2\nu(x') \wedge \nabla_{x'} \wedge \int_{\partial D} G_\tau(x', y') \tilde{\mathbf{a}}(y') ds_{y'}, \end{aligned} \quad (3.23)$$

where

$$G_\tau(x', y') = \frac{1}{4\pi} \frac{e^{i\tau\omega|x'-y'|}}{|x'-y'|} = G_0(x', y') + \frac{i\tau\omega}{4\pi} + \tau^2 \mathcal{R}(x', y').$$

It is easily verified that the remainder term $\mathcal{R}(x', y')$ satisfies

$$|\mathcal{R}(x', y')| = \mathcal{O}(|x' - y'|), \quad x', y' \in \partial D. \quad (3.24)$$

Hence, we have the following splitting

$$\begin{aligned} & 2\nu(x') \wedge \nabla_{x'} \wedge \int_{\partial D} G_\tau(x', y') \tilde{\mathbf{a}}(y') \, ds_{y'} \\ &= 2\nu(x') \wedge \nabla_{x'} \wedge \int_{\partial D} G_0(x', y') \tilde{\mathbf{a}}(y') \, ds'_y \\ & \quad + 2\tau^2 \nu(x') \wedge \nabla_{x'} \wedge \int_{\partial D} \mathcal{R}(x', y') \tilde{\mathbf{a}}(y') \, ds_{y'} \\ &= (\mathbf{M}_{\partial D}^0 \tilde{\mathbf{a}})(x') + (\mathcal{R}\tilde{\mathbf{a}})(x'). \end{aligned} \quad (3.25)$$

By using the mapping properties of layer potential operators in [35], it is straightforward to show that

$$\begin{aligned} & \|\mathcal{R}\tilde{\mathbf{a}}(\cdot)\|_{TH^{-1/2}(\partial D)} \\ & \leq C\tau^2 \|\tilde{\mathbf{a}}(\cdot)\|_{TH^{-1/2}(\partial D)} = C\tau^2 \|\mathbf{a}(\tau \cdot)\|_{TH^{-1/2}(\partial D)}, \end{aligned} \quad (3.26)$$

where C is a generic constant depending only on D and ω . Then, using again the change of variables in (3.22) to the integral equation (3.20), and the splitting (3.25), one has by direct calculations that

$$[I + M_{\partial D}^0 + \mathcal{R}]\tilde{\mathbf{a}}(x') = 2\varphi(\tau x'), \quad x' \in \partial D. \quad (3.27)$$

Next, we shall show

$$I + M_{\partial D}^0 \text{ is invertible from } TH^{-1/2}(\partial D) \text{ to } TH^{-1/2}(\partial D), \quad (3.28)$$

which together with (3.24) and (3.27) that

$$\tilde{\mathbf{a}}(x') = 2[(I + M_{\partial D}^0)^{-1} + \mathcal{O}(\tau^2)](\varphi(\tau \cdot))(x'), \quad (3.29)$$

thus proving the claim in (3.21). In order to show (3.28), we first note that $M_{\partial D}^0$ is an integral operator of order -1 , i.e., it is continuous from $TH^s(\partial D)$ to $TH^{s+1}(\partial D)$ (see [35], pp. 242). Hence, by the Reisz-Fredholm theory, it is sufficient to show that

$$(I + M_{\partial D}^0)\mathbf{b} = 0, \quad \mathbf{b} \in TH^{-1/2}(\partial D) \quad (3.30)$$

has only trivial solution, i.e., $\mathbf{b} = 0$. By using the fact that $M_{\partial D}^0$ is of degree -1 , we see that the spectrum of $M_{\partial D}^0$ is the same in $TH^{-1/2}(\partial D)$ and $C(\partial D)$. Then by Theorem 5.4 in [10], one must have $\mathbf{b} = 0$ in (3.30), which readily proves (3.28).

Finally, using (3.21) and the integral representations (3.18) and (3.19), it is by direct calculations to show (3.13) and (3.14).

The proof is completed. \square

Proof of Lemma 3.2. Since U_τ is smooth on $\partial\Omega$, we know both \tilde{U}_τ and \tilde{V}_τ are strong solutions which belong to $(C^1(\Omega \setminus \overline{D}_\tau) \cap C^{0,\alpha}(\overline{\Omega} \setminus D_\tau))^3$ with $0 < \alpha < 1$ (see [10, 11]). Hence, in the sequel, we shall work within the classic setting. To that end, we introduce $T(\Gamma)$, the spaces of all continuous tangential fields

\mathbf{a} equipped with the supremum norm, and $T^{0,\alpha}(\Gamma)$, the space of all Hölder continuous tangential fields equipped with the usual Hölder norm. We also need to introduce normed spaces of tangential fields possessing a surface divergence by

$$T_d(\Gamma) := \{\mathbf{a} \in T(\Gamma) | \text{Div } \mathbf{a} \in C(\Gamma)\}$$

and

$$T_d^{0,\alpha}(\Gamma) := \{\mathbf{a} \in T^{0,\alpha} | \text{Div } \mathbf{a} \in C^{0,\alpha}(\Gamma)\}$$

equipped with the norms

$$\|\mathbf{a}\|_{T_d} := \|\mathbf{a}\|_\infty + \|\text{Div } \mathbf{a}\|_\infty, \quad \|a\|_{T_d^{0,\alpha}} := \|a\|_{0,\alpha} + \|\text{Div } \mathbf{a}\|_{0,\alpha}.$$

We again employ the boundary layer potential operators introduced in (3.16) and (3.17), and refer to [10] and [11] for mapping and jumping properties in the classical setting.

Similar to the proof of Proposition 3.2, instead of proving the uniform well-posedness of the Maxwell equations (3.5), we focus on deriving the desired estimate (3.8). However, it should be pointed out that the unique existence of strong solutions to (3.5) for sufficiently small τ can be directly seen from our subsequent argument.

By the Stratton-Chu formula, we have (see [11], Theorem 6.2)

$$\begin{aligned} \tilde{V}_\tau(x) = & -\nabla_x \wedge \int_{\partial\Omega} \nu(y) \wedge \tilde{V}_\tau(y) G(x, y) \, ds_y \\ & + \nabla_x \wedge \int_{\partial D_\tau} \nu(y) \wedge \tilde{V}_\tau(y) G(x, y) \, ds_y \\ & - \frac{1}{i\omega} \nabla_x \wedge \nabla_x \wedge \int_{\partial\Omega} \nu(y) \wedge U_\tau(y) G(x, y) \, ds_y, \quad x \in \Omega \setminus \overline{D}_\tau. \end{aligned} \quad (3.31)$$

Set

$$\begin{aligned} \mathbf{a}_1(x) &= \nu(x) \wedge \tilde{V}_\tau(x), \quad x \in \partial\Omega, \\ \mathbf{a}_2(x) &= \nu(x) \wedge \tilde{V}_\tau(x), \quad x \in \partial D_\tau, \end{aligned}$$

and

$$\begin{aligned} P_1(x) &= -\frac{1}{i\omega} \nu(x) \wedge \nabla_x \wedge \nabla_x \wedge \int_{\partial\Omega} \nu(y) \wedge U_\tau(y) G(x, y) \, ds_y, \quad x \in \partial\Omega, \\ P_2(x) &= -\frac{1}{i\omega} \nu(x) \wedge \nabla_x \wedge \nabla_x \wedge \int_{\partial\Omega} \nu(y) \wedge U_\tau(y) G(x, y) \, ds_y, \quad x \in \partial D_\tau \end{aligned}$$

By letting x approach ∂D_τ^+ , and using the mapping properties of $\mathbf{M}_{\partial D_\tau}$ and $\mathbf{M}_{\partial\Omega}$, one has

$$\begin{aligned} (I + \mathbf{M}_{\partial\Omega})\mathbf{a}_1(x) - (\mathbf{M}_{\partial D_\tau}\mathbf{a}_2)(x) &= 2P_1(x), \quad x \in \partial\Omega, \\ (I - \mathbf{M}_{\partial D_\tau})\mathbf{a}_2(x) + (\mathbf{M}_{\partial\Omega}\mathbf{a}_1)(x) &= 2P_2(x), \quad x \in \partial D_\tau. \end{aligned} \quad (3.32)$$

By using a similar scaling argument to that in the proof of Proposition 3.2, one can show

$$(\mathbf{M}_{\partial D_\tau}\mathbf{a}_2)(\tau x') = (\mathbf{M}_{\partial D}^0 \tilde{\mathbf{a}}_2)(x') + (\mathcal{R}\tilde{\mathbf{a}}_2)(x'), \quad x' \in \partial D, \quad (3.33)$$

where $\tilde{\mathbf{a}}_2(x') := \mathbf{a}_2(\tau x')$ for $x' \in \partial D$, and

$$\|\mathcal{R}\tilde{\mathbf{a}}_2\|_{T_d^{0,\alpha}(\partial D)} \leq C\tau^2 \|\tilde{\mathbf{a}}_2\|_{T_d^{0,\alpha}(\partial D)}. \quad (3.34)$$

By setting $\tilde{P}_2(x') = P_2(\tau x')$ for $x' \in \partial D$, the system of integral equations (3.32) can be further formulated as

$$\begin{aligned} (I + \mathbf{M}_{\partial\Omega})\mathbf{a}_1(x) - (\widetilde{\mathbf{M}}_{\partial D}\tilde{\mathbf{a}}_2)(x) &= 2P_1(x), \quad x \in \partial\Omega, \\ (I - M_{\partial D}^0)\tilde{\mathbf{a}}_2(x') - (\mathcal{R}\tilde{\mathbf{a}}_2)(x') + (\mathbf{M}_{\partial\Omega}\mathbf{a}_1)(\tau x') &= 2\tilde{P}_2(x'), \quad x' \in \partial D, \end{aligned} \quad (3.35)$$

where

$$(\widetilde{\mathbf{M}}_{\partial D}\tilde{\mathbf{a}}_2)(x) = (\mathbf{M}_{\partial D\tau}\mathbf{a}_2)(x), \quad x \in \partial\Omega,$$

obtained by replacing the variable y in defining $\mathbf{M}_{\partial D\tau}\mathbf{a}_2$ (cf. (3.16)) by $\tau y'$ with $y' \in \partial D$. Since the integral kernel for $(\widetilde{\mathbf{M}}_{\partial D}\tilde{\mathbf{a}}_2)(x)$ is smooth when $x \in \partial\Omega$, it is straightforwardly shown that

$$\|\widetilde{\mathbf{M}}_{\partial D}\tilde{\mathbf{a}}_2\|_{T_d^{0,\alpha}(\partial\Omega)} \leq C\tau^2 \|\tilde{\mathbf{a}}_2\|_{T_d^{0,\alpha}(\partial D)}. \quad (3.36)$$

Similarly, one can see that

$$\|\mathbf{M}_{\partial\Omega}\mathbf{a}_1(\tau \cdot)\|_{T_d^{0,\alpha}(\partial D)} \leq C\|\mathbf{a}_1\|_{T_d^{0,\alpha}(\partial\Omega)}. \quad (3.37)$$

Define

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}_1 \\ \tilde{\mathbf{a}}_2 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 2P_1 \\ 2\tilde{P}_2 \end{pmatrix}, \quad (3.38)$$

and

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_{11} & \mathbf{0} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} \mathbf{0} & \mathbf{R}_{12} \\ \mathbf{0} & \mathbf{R}_{22} \end{pmatrix}, \quad (3.39)$$

where

$$\mathbf{L}_{11} := I + \mathbf{M}_{\partial\Omega}, \quad \mathbf{L}_{21} := \mathbf{M}_{\partial\Omega}, \quad \mathbf{L}_{22} := I - M_{\partial D}^0$$

and

$$\mathbf{R}_{12} := \widetilde{\mathbf{M}}, \quad \mathbf{R}_{22} := \mathcal{R}.$$

Then the system of integral equations (3.35) can be written as

$$(\mathbf{L} - \mathbf{R})\mathbf{a} = \mathbf{P}. \quad (3.40)$$

\mathbf{L}_{11} is compact from $T_d^{0,\alpha}(\partial\Omega)$ to $T_d^{0,\alpha}(\partial D)$ and \mathbf{L}_{22} is compact from $T_d^{0,\alpha}(\partial D)$ to $T_d^{0,\alpha}(\partial D)$ (cf. [11], Chapter 6). Since ω is not an EM eigenvalue to (1.13), by the Riesz-Fredholm theory, \mathbf{L}_{11} is invertible (cf. [10], Theorem 4.23), and \mathbf{L}_{22} is also invertible (cf. [10], Theorem 5.4). Hence, it is straightforward to verify that \mathbf{L} is invertible from $T_d^{0,\alpha}(\partial\Omega) \wedge T_d^{0,\alpha}(\partial D)$ to itself, and

$$\mathbf{L}^{-1} = \begin{pmatrix} \mathbf{L}_{11}^{-1} & \mathbf{0} \\ -\mathbf{L}_{22}^{-1}\mathbf{L}_{21}\mathbf{L}_{11}^{-1} & \mathbf{L}_{22}^{-1} \end{pmatrix}.$$

Hence, by noting (3.34) and (3.36), we see that for sufficiently small τ ,

$$\mathbf{a} = (\mathbf{L} - \mathbf{R})^{-1}\mathbf{P}. \quad (3.41)$$

Finally, by using the mapping property of the electric dipole operator (see Theorem 6.17 in [11]), one can show

$$\|P_1\|_{T_d^{0,\alpha}(\partial\Omega)} \leq C\|U_\tau\|_{(C^2(\partial\Omega))^3},$$

which together with Proposition 3.2 further implies that

$$\|P_1\|_{T_d^{0,\alpha}(\partial\Omega)} \leq C \left(\tau^3 \|\psi\|_{TH_{Div}^{-1/2}(\partial\Omega)} + \tau^2 \|\varphi(\tau \cdot)\|_{TH^{-1/2}(\partial D)} \right). \quad (3.42)$$

Moreover, it is straightforward to verify that

$$\|\tilde{P}_2\|_{T_d^{0,\alpha}(\partial D)} \leq C\|U_\tau\|_{(C^2(\partial\Omega))^3}$$

and hence

$$\|\tilde{P}_2\|_{T_d^{0,\alpha}(\partial D)} \leq C \left(\tau^3 \|\psi\|_{TH_{Div}^{-1/2}(\partial\Omega)} + \tau^2 \|\varphi(\tau \cdot)\|_{TH^{-1/2}(\partial D)} \right). \quad (3.43)$$

By combining (3.41), (3.42) and (3.43), one readily has

$$\|\mathbf{a}_1\|_{T_d^{0,\alpha}(\partial\Omega)} \leq C \left(\tau^3 \|\psi\|_{TH_{Div}^{-1/2}(\partial\Omega)} + \tau^2 \|\varphi(\tau \cdot)\|_{TH^{-1/2}(\partial D)} \right),$$

which immediately implies (3.8).

The proof of Lemma 3.2 is complete. \square

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